§ Motivation

Have seen: $\mathcal{M}_n : S \longmapsto \{(E,\alpha)/S\}/\cong$ $\quad n \geq 3$

is representable when restricted to loc noeth. S.

Reason Whole theory relied on theory of line bundles on ECs, which relies on cohom & bc, which is only proved for noetherian base.

(Loc free sheaves are not nec coherent in non-noetherian situation.)

Today R rg, $(E,\alpha)/R$. Then $\exists$ noetherian $R_0 \to R$ + $(E_0,\alpha_0)/R_0$ s.th. $(E,\alpha) \cong R \otimes_{R_0} (E_0,\alpha_0)$.

Cor M_n represents $\{(E,\alpha)\}/\cong$ on all schemes.

Reference for following [EGA IV_3, IV_4]

§ Setting

$(I, \leq)$ directed set : $\forall \lambda_1, \lambda_2 \ \exists \mu$ s.th. $\lambda_1, \lambda_2 \leq \mu$.

$(R_\lambda)_{\lambda \in I}$ + $(R_\lambda \xrightarrow{\rho_{\mu\lambda}} R_\mu)_{\lambda \leq \mu}$

s.th. $\rho_{\nu\mu} \circ \rho_{\mu\lambda} = \rho_{\nu\lambda}$ direct system of rings.

$$R := \operatorname{colim}_{\lambda \in I} R_\lambda$$

Interesting cases

1) R ring, $\mathfrak{p} \in \operatorname{Spec} R$ $\quad I = \{ f \in R \setminus \mathfrak{p} \}$

ordered by $f \leq g \iff f \mid g$.

Then $R_\mathfrak{p} = \operatorname{colim}_{f \in I} R[f^{-1}]$

2) R ring $\quad I = \{$ finite subsets $S \subseteq R \}$

ordered by inclusion

Then $R = \operatorname{colim}_{S \in I}$ ($\mathbb{Z}$-algebra generated by $S \subseteq R$)

$= \operatorname{Im}(\mathbb{Z}[T_s \,;\, s \in S] \longrightarrow R)$

Slogan Every ring is filtered colimit of finite type $\mathbb{Z}$-alg.

3) (Variant) K field, k $\mathbb{F}_p$ or $\mathbb{Q}$ its prime field

$I = \{ K_0 \subseteq K, \ K_0/k \text{ fin gen} \}$

ordered by inclusion.

Then $K = \operatorname{colim}_{K_0 \in I} K_0$.

Consider C_λ categories of <u>objects of interest</u> $/R_\lambda$

<u>Objects of interest</u> schemes, diagrams of schemes, group schemes, ECs, schemes X_λ + qcoh $\mathcal{O}_{X_\lambda}$-module, ...

Required to possess base extension functor

$$C_\lambda \to C_\mu, \quad X_\lambda \mapsto R_\mu \otimes_{R_\lambda} X_\lambda \qquad \lambda \le \mu$$

+ natural isomorphisms

$$R_\nu \otimes_{R_\mu} (R_\mu \otimes_{R_\lambda} X_\lambda) \xrightarrow{\ \cong\ } R_\nu \otimes_{R_\lambda} X_\lambda$$

Similarly need C = such objects $/R$

+ $R \otimes_{R_\lambda} - : C_\lambda \to C$.

Get $\operatorname{colim}_{\lambda \in I} C_\lambda \xrightarrow{\ \Phi\ } C$

<u>Here</u> objects of $\operatorname{colim}_{\lambda\in I} C_\lambda \underset{\text{def}}{=} \coprod_\lambda \operatorname{Obj}(C_\lambda)$

$$\operatorname{Hom}(X_\lambda, Y_\mu) \underset{\text{def}}{=} \operatorname{colim}_{\nu\geq\lambda,\mu} \operatorname{Hom}(R_\nu \otimes_{R_\lambda} X_\lambda, R_\nu \otimes_{R_\mu} Y_\mu)$$

1) Φ full means:

$:= R \otimes_{R_\lambda} X_\lambda$

Given X_λ, Y_λ + $f: X \longrightarrow Y$,

$\exists\ \mu\geq\lambda$ + $f_\mu: X_\mu \longrightarrow Y_\mu$ s.th. $f = R \otimes_{R_\mu} f_\mu$.

2) Φ faithful:

Given $f_\lambda, g_\lambda: X_\lambda \longrightarrow Y_\lambda$ s.th. $f = g$

$\exists\ \mu\geq\lambda$ s.th. $f_\mu = g_\mu$.

3) Φ Essentially surjective:

Given X, $\exists\ \lambda$ + X_λ s.th. $X \cong R \otimes_{R_\lambda} X_\lambda$.

§ Approximation for affines

<u>Thm</u> (EGA IV_3 § 8.8) $\mathcal{C}_\lambda$:= finitely presented R_λ-algebras

($\mathcal{C}$ such R-algebras)

Then $\operatorname{colim}_\lambda \mathcal{C}_\lambda \to \mathcal{C}$ equivalence.

<u>Full</u> $A_\lambda, B_\lambda / R_\lambda$. Write $A_\lambda = R_\lambda[T_1, \dots, T_n] / f_1, \dots, f_r$

$B_\lambda = R_\lambda[S_1, \dots, S_m] / g_1, \dots, g_s$.

$\varphi : A \to B$ given by $\varphi(T_i) \in R[S_1, \dots, S_m]$

s.th. $f_i(\varphi(T_1), \dots, \varphi(T_n)) \in (g_1, \dots, g_s)$

i.e. may be written as $\sum_{j=1}^{s} p_{ij} \cdot g_j$ in $R[S_1, \dots, S_m]$

$\exists \mu > \lambda$ s.th. all coefficients of all $\varphi(T_i)$

+ all coeff of all p_{ij} $\in R_\mu$.

$\Rightarrow$ May define $\varphi_\mu : A_\mu \to B_\mu$.

<u>Faithful</u> Assume $\varphi_\lambda, \psi_\lambda : A_\lambda \to B_\lambda$ s.th. $\varphi = \psi$.

<u>Means</u> $\varphi_\lambda(T_i) - \psi_\lambda(T_i) = \sum_{j=1}^{s} p_{ij} \cdot g_j$

$\exists p_{ij} \in R[S_1, \dots, S_m]$ s.th.

$\exists\, \mu \geq \lambda$ s.th. all coeff of all $p_{ij} \in R_\mu$.

$\Longrightarrow \quad \varphi_\mu = \psi_\mu$.

Essentially surjective Given A/R,

write $A \cong R[T_1, \dots, T_n]/_{f_1, \dots, f_m}$.

$\exists\, \lambda$ s.th. all coeff of all $f_i \in R_\lambda$

$\Longrightarrow \exists\, A_\lambda$ s.th. $A \cong R \otimes_{R_\lambda} A_\lambda$. $\square$

Properties Let P be any of following:

1) Open immersion

2) Closed immersion

3) surjective

4) finite

Given $\varphi_\lambda : A_\lambda \to B_\lambda$,

$\varphi : A \to B$ has P

$\Leftrightarrow$ φ_μ has P for some $\mu \geq \lambda$.

Note Direction $\Leftarrow$ always clear.

Proof of 1): $\operatorname{Spec} B \xrightarrow{\varphi} \operatorname{Spec} A$ open immersion.

$\operatorname{Spec} B$ qc $\Longrightarrow \exists$ fin many f_i s.th. $\operatorname{Spec} B = \bigcup D(f_i)$

$\exists\ \mu \geq \lambda$ s.th. $f_i \in A_\mu$

+ $A_\mu[f_i^{-1}] \xrightarrow{\cong} B_\mu[f_i^{-1}]$
+ $A_\mu \longrightarrow \prod_i A_\mu[f_i^{-1}]$ factors through B_μ

$\Bigg\}$ by fully faithfulness

$\Longrightarrow \operatorname{Spec} B_\mu \cong \bigcup D(f_i) \subseteq \operatorname{Spec} A_\mu.$

<u>Example</u> $I = \mathbb{Z}_{\geq 1}$, $R_n = k[\![t]\!][t^{1/n}]$

$R = \operatorname{colim}_{n|m} R_n.$

Then $A = R/(t, t^{1/2}, t^{1/3}, \ldots)$ is a non-finite presentation R-algebra s.th. $\nexists\ n$ + A_n/R_n with $A = R \otimes_{R_n} A_n$.

(Exercise)

Now consider $C_\lambda = \{(A_\lambda, M_\lambda)/R_\lambda\}$

A_λ fin pres R_λ-alg

M_λ fin pres A_λ-module.

Morphisms:

$$\mathrm{Hom}((A_\lambda, M_\lambda), (B_\lambda, N_\lambda)) =$$

$$\left\{ \varphi : A_\lambda \to B_\lambda \; + \; A_\lambda\text{-linear } M_\lambda \to N_\lambda \atop \longleftrightarrow B_\lambda \text{ linear } B_\lambda \otimes_{\varphi, A_\lambda} M_\lambda \to N_\lambda \right\}$$

<u>Thm</u> (EGA IV_3 §8.5)

$$\operatorname{colim}_{\lambda \in I} C_\lambda \to C \quad \text{is an equivalence.}$$

Moreover, given $M_\lambda \to N_\lambda \to Q_\lambda \to 0 \;/ A_\lambda$,

$M \to N \to Q \to 0$ exact $\iff \exists\, \mu \geq \lambda$ s.t.

$$M_\mu \to N_\mu \to Q_\mu \to 0 \text{ exact.}$$

(This means cokernels may be approximated.)

both finite presentation

<u>Prop</u> Given (A_λ, M_λ), the following are equivalent

1) M finite loc free over A

2) $\exists\, \mu \geq \lambda$ s.th. M_μ finite loc free over A_μ.

<u>Proof</u> 2) $\Rightarrow$ 1) trivial.

1) $\Rightarrow$ 2): Pick $\operatorname{Spec} A = \bigcup D(f_i)$ s.th.

$M[f_i^{-1}]$ free over $A[f_i^{-1}]$.

$\exists\, \mu \geq \lambda$ s.th. f_i lie in A_μ. $\Rightarrow$ $M_\mu[f_i^{-1}] / A_\mu[f_i^{-1}]$ defined.

Fully faithfulness from Thm:

Given $A[f_i^{-1}]^{\oplus r_i} \cong M[f_i^{-1}]$, $\exists\, \nu \geq \mu$ s.th.

comes from iso $A_\nu[f_i^{-1}]^{\oplus r_i} \cong M_\nu[f_i^{-1}]$. $\square$

<u>Variant</u> Given 1), pick $A^{\oplus r} \xrightarrow{\varphi} M \to 0$

Has section $s: M \to A^{\oplus r}$ by assumption.

Thm $\Rightarrow$ $\exists\, \mu$ s.th. 1) $\varphi: A_\mu^{\oplus r} \to M_\mu$ defined

2) φ surjective 3) s also defined 4) $\varphi \circ s = \mathrm{id}$. $\square$

§ Globalization $\quad S_\lambda = \operatorname{Spec} R_\lambda, \quad S = \operatorname{Spec} R$

Recall $X \to S$ finite presentation $\overset{=}{\text{def}}$

$X = U_1 \cup \cdots \cup U_n \quad U_i = \operatorname{Spec} A_i \quad A_i$ fin pres $/R$

$+ \; X$ quasi-separated (!)

$\Longrightarrow \quad U_i \cap U_j$ quasi-compact $\forall i,j$

i.e. $\quad \cong \bigcup_{\text{finite}} D(f_{ijk}) \subseteq U_i$

$\Longrightarrow$ finite presentation schemes$/R$

$\cong$ certain finite diagrams of finite presentation R-algebras.

Our affine statements generalize to:

Thm $\quad \operatorname{colim}_{\lambda \in I} \{ \text{fin pres } X_\lambda / S_\lambda \} \xrightarrow{\cong} \{ \text{fin pres } X/S \}$

$\operatorname{colim}_{\lambda \in I} \{ X_\lambda / S_\lambda + \mathcal{F}_\lambda \text{ q coh fin pres } \mathcal{O}_{X_\lambda}\text{-mod} \}$

$\xrightarrow{\cong} \{ X/S + \mathcal{F} \text{ q coh fin pres } \mathcal{O}_X\text{-mod} \}$

§ Properties of morphisms (see EGA IV_3 §8.10)

Thm $f_\lambda : X_\lambda \longrightarrow Y_\lambda$ map of fin. pres. S_λ-schemes.

Then f has $P \iff \exists \mu \geq \lambda$ s.th. f_μ has P for:

$P \in \{$ isomorphism,

monomorphism,

(open resp. closed) immersion,

separated,

surjective,

radiciel ($\forall$ fields k, $X(k) \hookrightarrow Y(k)$),

affine,

quasi-affine,

finite,

quasi-finite,

proper,

projective,

quasi-projective $\}$

EGA I §5.6

noeth. reduction

On the properness

Thm (Chow's Lemma, EGA IV_3 8.10.5.1)

Given $X, Y/S$ finite presentation and $X \to Y$ separated. There exists a diagram

$$\begin{tikzcd} X' \arrow[r, "\text{immersion}"] \arrow[d, "\text{projective + surjective}"'] & P \arrow[d, "\text{projective}"] \\ X \arrow[r] & Y \end{tikzcd}$$

Idea After noetherian reduction, may assume X irreducible. Write $X = \bigcup_{i=1}^{r} U_i$ open affine covering.

U_i affine $\Rightarrow$ $\exists$ immersion $U_i \hookrightarrow \mathbb{P}_Y^{n_i}$.

Obtain $\bigcap_i U_i =: U \xrightarrow{\psi} P := \mathbb{P}_Y^{n_1} \times_Y \cdots \times_Y \mathbb{P}_Y^{n_r}$.

Then $X' :=$ closure of $\Gamma_\psi \subseteq X \times_Y P$ has required properties.

Now let $X_\lambda \xrightarrow{f_\lambda} Y_\lambda$ be s.th. $X \xrightarrow{f} Y$ proper.
(separated)

Pick diagram as in Chow's Lemma.

$$\begin{tikzcd} X'_\lambda \arrow[r, "j_\lambda"] \arrow[d, "g_\lambda"'] & P_\lambda \arrow[d, "p_\lambda"] \\ X_\lambda \arrow[r, "f_\lambda"'] & Y_\lambda \end{tikzcd}$$

Then $X' \xrightarrow{f \circ g} Y$ proper, so $X' \xrightarrow{j} P$ proper.

$\Longrightarrow$ $X' \xrightarrow{j} P$ closed immersion

$\Longrightarrow$ $\exists \mu \geq \lambda$ s.th. j_μ closed immersion.

$\Longrightarrow$ $p_\mu \circ j_\mu = f_\mu \circ g_\mu$ proper

$\Longrightarrow$ f_μ proper. $\square$

g_μ surjective + f_μ separated

On smoothness

Thm (EGA IV_3 11.2.6)

$f_\lambda : X_\lambda \longrightarrow Y_\lambda$ map of fin pres S_λ-schemes.

$\mathcal{F}_\lambda$ qcoh $\mathcal{O}_{X_\lambda}$-module.

Then $\mathcal{F}$ flat over $Y \Leftrightarrow \exists\, \mu \geq \lambda$ s.th.

$\mathcal{F}_\mu$ flat over Y_μ

Cor (EGA IV_4 17.7.8)

$f_\lambda : X_\lambda \longrightarrow Y_\lambda$ as above.

f smooth $\Leftrightarrow \exists\, \mu \geq \lambda$ s.th. f_μ smooth.

Proof f smooth $\xRightarrow{\text{above Thm}}$ $\exists\, \mu \geq \lambda$ s.th. f_μ flat

Now $\Omega^1_{X/Y} = \mathcal{O}_X \underset{\mathcal{O}_{X_\mu}}{\otimes} \Omega^1_{X_\mu/Y_\mu}$ loc free

of correct rank $\underset{\text{prev. Prop.}}{\Longrightarrow}$ $\exists\, \nu \geq \mu$ s.th. $\Omega^1_{X_\nu/Y_\nu}$

loc free.

$\square$

Cor Equivalent: $X \to \operatorname{Spec} A$ finite presentation + smooth

$\iff \exists$ finite type $\mathbb{Z}$-algebra A_0 + $A_0 \to A$

+ X_0 / A_0 fin. pres. smooth s.th. $X \cong A \otimes_{A_0} X_0$.

§ Application to abelian schemes

Prop $(X, m)/S = \operatorname{Spec} R$ abelian scheme.

Then $\exists$ fin. type $\mathbb{Z}$-alg R_0 + $(X_0, m_0)/R_0$ s.th.

s.th. $(X, m) \cong R \otimes_{R_0} (X_0, m_0)$.

Proof Write $R = \operatorname{colim}_{\substack{R' \subseteq R \\ \text{finite type } \mathbb{Z}\text{-alg}}} R'$

By above, $\exists$ R' + $(X', m')/R'$

X' proper smooth

$m' : X' \times_{R'} X' \to X'$ group scheme structure.

Only issue: fibers of $X' \to \operatorname{Spec} R'$ might be non-connected.

Prop (Stacks [0E0N])

$f : X \to S$ proper flat fin. pres.

+ fibers geometrically reduced.

geometric fiber

Then $n_{X/S} : S \to \mathbb{Z}$, $s \mapsto \# \pi_0(X(\bar{s}))$

is locally constant.

Application: Define $R' \to R_0$ s.th.

$$\operatorname{Spec} R_0 = \bigcup_{\substack{C \in \pi_0(\operatorname{Spec} R') \\ C \cap \operatorname{Im}(\operatorname{Spec} R \to \operatorname{Spec} R') \neq \emptyset}} C$$

(R' noetherian $\Rightarrow$ $\pi_0(\operatorname{Spec} R')$ finite)

Then $(X_0, m_0) := R_0 \otimes_{R'} (X', m')$

is abelian scheme as requested. $\square$:)

Rmk $n_{X|S}$ is usually lower semi-continuous.

Consider branched double cover $\mathbb{A}^1_{\mathbb{C}} \xrightarrow{t \mapsto t^2} \mathbb{A}^1_{\mathbb{C}}$.

Then $$n_{\mathbb{A}^1_{\mathbb{C}} \to \mathbb{A}^1_{\mathbb{C}}}(x) = \begin{cases} 2 & x \neq 0 \\ 1 & x = 0 \end{cases}.$$

§ Application to cohom & bc

$S = \operatorname{Spec} A$

<u>Thm</u> $X \to S$ proper + fin pres.

$\mathcal{F}$ fin pres qcoh $\mathcal{O}_X$-mod, flat $/S$

Then $\exists$ perfect complex $K^\bullet$ of A-modules

s.th. $H^i(B \otimes_A \mathcal{F}) = H^i(B \otimes_A K^\bullet) \quad \forall A \to B.$

<u>Proof</u> $\exists$ noetherian A_0 + $X_0 \to S_0 = \operatorname{Spec} A_0$ proper

+ $\mathcal{F}_0$ fin pres qcoh $\mathcal{O}_{X_0}$-mod, flat $/\mathcal{O}_{S_0}$

s.th. $(X, \mathcal{F}) \cong S \times_{S_0} (X_0, \mathcal{F}_0)$

A_0 noetherian $\Longrightarrow$ X_0 noetherian

$\Longrightarrow$ $\mathcal{F}_0$ coherent.

Usual cohom & bc

$\Longrightarrow$ $\exists$ perfect $K_0^\bullet$ of A_0-modules s.th.

$\forall A_0 \to B, \quad H^i(B \otimes_{A_0} \mathcal{F}_0) = H^i(B \otimes_{A_0} K_0^\bullet)$

Hence may put $K^\bullet = K_0^\bullet$. $\square$

Cor E/S EC, $\mathcal{L}$ lb on E

If $\deg \mathcal{L}(s) = d \geq 1$ $\forall s$, $p_* \mathcal{L}$ vb of rank d.

In partic, $E \hookrightarrow \mathbb{P}(p_* \mathcal{O}(3e))$ as cubic curve.

Cor $E \xrightarrow{\cong} \mathrm{Pic}^0_{E/S}$ also for non-noeth. S

(resp. as functor on all schemes, not just noetherian ones.)